

Critical dynamic events at the crisis of transition to spatiotemporal chaosKaifen He^{1,2,3} and Abraham C.-L. Chian^{3,4}¹CCAST (World Laboratory), Beijing, 100080 China²Institute of Low Energy Nuclear Physics, Beijing Normal University, Beijing, 100875 China³World Institute for Space Environment Research (WISER), NITP, University of Adelaide, Adelaide, SA 5005, Australia⁴National Institute for Space Research (INPE), P.O. Box 515, São José dos Campos, SP 12227-010, Brazil

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In the driven/damped drift-wave plasma system a collision of the weak chaotic attractor with a saddle point is demonstrated at the crisis that induces a transition from a spatially coherent state to spatiotemporal chaos (STC). The phenomenon of the collision is consistent with the previous observation of the ‘pattern resonance’ that triggers the crisis. Subsequent to the collision, before the system is ejected to the STC attractor, there is evidence of another critical dynamic event involving state transition of a mode phase. The second event plays a crucial role in the destruction of the spatial coherence.

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I. INTRODUCTION

Turbulence is a very common phenomenon in fluids, plasmas, optics, etc. A fully developed turbulence is not only chaotic in time, but also erratic in space. The investigation of the mechanism for the onset of turbulence has attracted much attention for decades. As is well known, Landau’s picture of turbulence is a sequence of stepwise increases in the number of frequencies. Later on, it was realized that three incommensurate frequencies directly lead to chaos. This Ruelle-Takens route has been observed theoretically in the models of wave-wave interaction [1,2]. In a drift-wave experiment it was demonstrated that the Ruelle-Takens route leads to weak turbulence [3,4]. In contrast to fully developed turbulence where a wave is broken, in weak turbulence the spatial behavior remains coherent. In these examples a sequence of local bifurcations is not sufficient to destroy the spatial coherence; it cannot explain wavebreaking in fully developed turbulence. If wavebreaking is not a result of local bifurcations, is it possible that a global bifurcation, i.e., a crisis, is responsible for it?

Crisis is a sudden change of a chaotic attractor, including the sudden creation and expansion of a chaotic attractor, a merger of two chaotic attractors, and the inverse of these processes [5–7], which has been widely studied in time-dependent systems. There are also examples of crisis in extended systems. For instance, in a model derived from the Kuramoto-Sivashinsky equation it is found that a high-dimensional interior crisis leads to an abrupt expansion of the chaotic attractor, which results in a sudden increase in the system chaoticity [8]. In this model, the spatial coherence still persists after the crisis. In contrast, there is an example that spatial coherence is destroyed by a crisis [9], described by the model of a driven/damped plasma drift-wave equation:

$$\frac{\partial \phi}{\partial t} + a \frac{\partial^3 \phi}{\partial t \partial x^2} + c \frac{\partial \phi}{\partial x} + f \phi \frac{\partial \phi}{\partial x} = -\gamma \phi - \epsilon \sin(x - \Omega t); \quad (1)$$

here $\phi(x + 2\pi) = \phi(x)$, and $a < 0, c, f$ are constants. In Ref.

[9] we have shown that for a given Ω in certain regime there exists a critical $\epsilon = \epsilon_c$, if $\epsilon < \epsilon_c$ the system dynamics is spatially regular (SR) although temporally it can be chaotic, while if $\epsilon > \epsilon_c$ a crisis occurs in the time evolution leading to a transition to spatiotemporal chaos (STC). For example, for $\Omega = 0.65$ we have $\epsilon_c \approx 0.20$.

In general, a crisis occurs when a chaotic attractor collides with an unstable periodic orbit. For example, if the orbit point “by chance” lands near a stable manifold segment of an unstable orbit of the saddle point, the orbit then moves towards this fixed point, following the direction of its stable manifold until being ejected to the unstable manifold and moving to another attractor [5]. Both crises observed in Refs. [8,9] are due to collision with the unstable orbit of a saddle point, respectively. In particular in Ref. [8] a collision with the saddle point was demonstrated at the critical transition parameter by projecting the orbit to low-dimensional phase space. It would then be interesting to see in system (1) whether and in what representation one can observe such a collision with the saddle point in the transition to the STC. Besides, after the crisis, in Ref. [8] the wave remains smooth but in Eq. (1) the wave is broken. The question is, then, in the latter case, is there any further dynamic event, other than a possible collision with the saddle point, that can destroy the spatial coherence? These are the questions we try to answer in the present work.

In Sec. II we demonstrate that a collision with the saddle point occurs at the critical parameter point for the transition of a weak chaotic attractor to the STC. In Sec. III the collision with the saddle point at the onset of STC is investigated. In Sec. IV it is found that just after the collision a subsequent dynamic event occurs, through which the phase of the master mode crosses a critical value and then experiences a transition in its dynamical behavior. The important role of this event in wavebreaking is addressed. Finally, Sec. V gives the conclusion and a discussion.

II. COLLISION WITH THE SADDLE POINT AND TRANSITION TO SPATIOTEMPORAL CHAOS

In the first place, we need to find out what a saddle point is like in Eq. (1), where the steady solutions are traveling

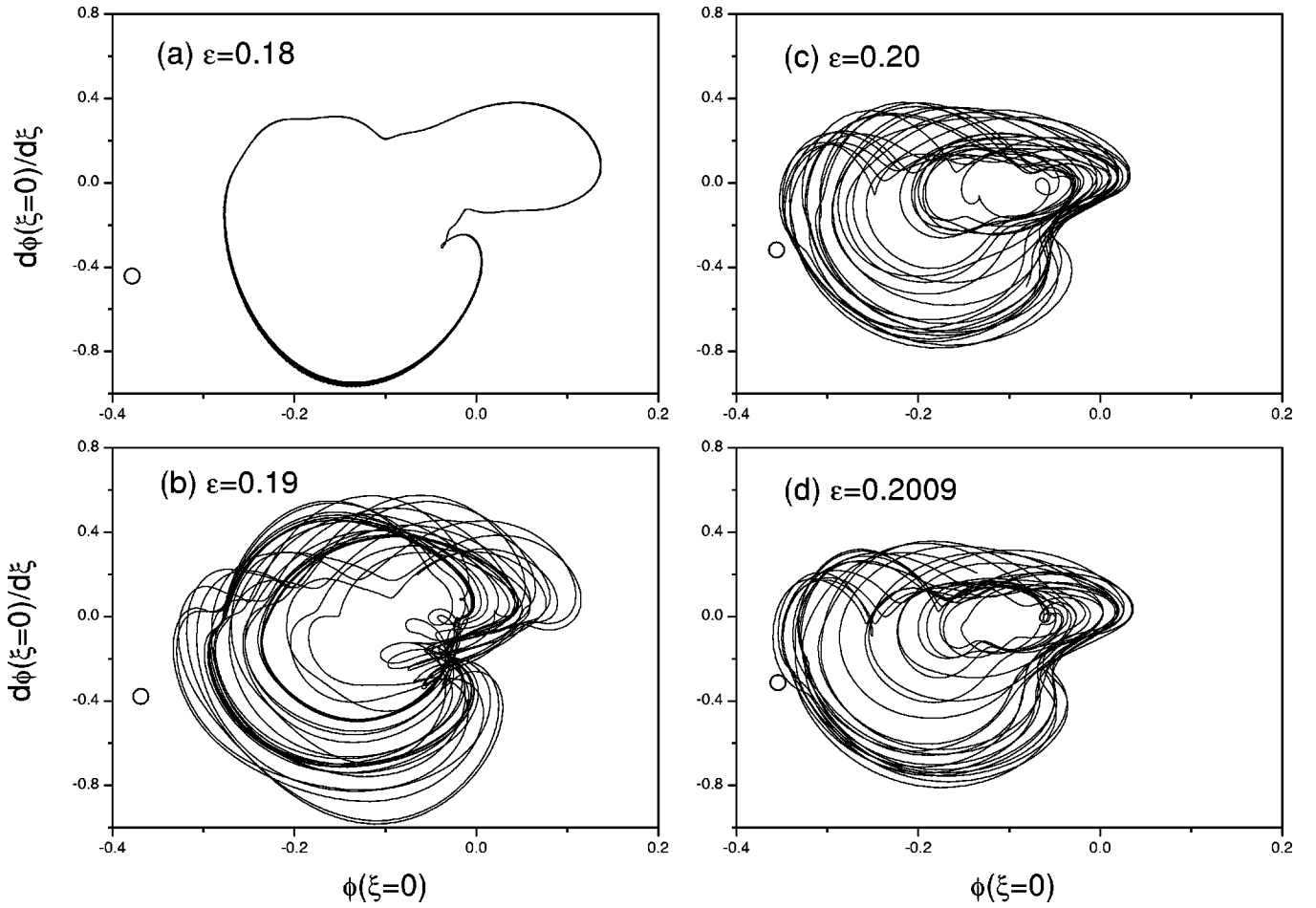


FIG. 1. Asymptotic SR attractor in the phase space $\phi(\xi=0)$ vs $\partial\phi(\xi=0)/\partial\xi$ for (a) $\epsilon=0.18$, (b) $\epsilon=0.19$, (c) $\epsilon=0.20$, and (d) $\epsilon=0.2009$, with $\Omega=0.65$. “○” shows the respective saddle points (SSW).

waves. Our investigation shows that it is a steady wave (SW) solution with a saddle instability [9]. For given parameters in the driver frame $\xi=x-\Omega t$, $\tau=t$, in general a SW $\phi_0(\xi)$ can be solved from $\partial\phi_0(\xi)/\partial\tau=0$ by expanding $\phi_0(\xi) \equiv \sum_{k=1}^{\infty} A_k \cos(k\xi + \theta_k)$. A SW $\phi_0(\xi)$ is a fixed point in the Fourier space, for the mode amplitudes $\{A_k\}$ and phases $\{\theta_k\}$ are all constants, respectively. When free dimensions at a fixed point are perturbed, if the complex conjugate eigenvalues in one dimension are degenerated to become real with positive and negative values, the fixed point is unstable to the saddle instability. A saddle steady wave (SSW) is a saddle point in the Fourier space, denoted as $\phi_0^*(\xi)$ henceforth. In the (ξ, τ) frame one can expect to see the phenomenon of collision with the saddle point at crisis.

By substituting $\phi(\xi, \tau) = \phi_0(\xi) + \delta\phi(\xi, \tau)$ into Eq. (1), the perturbation wave $\delta\phi(\xi, \tau)$ is governed by the following equation:

$$\begin{aligned} \frac{\partial}{\partial\tau} \left[1 + a \frac{\partial^2}{\partial\xi^2} \right] \delta\phi - \Omega \frac{\partial}{\partial\xi} \left[1 + a \frac{\partial^2}{\partial\xi^2} \right] \delta\phi + c \frac{\partial}{\partial\xi} \delta\phi + \gamma \delta\phi \\ + f \frac{\partial}{\partial\xi} [\phi_0(\xi) \delta\phi] + f \delta\phi \frac{\partial}{\partial\xi} \delta\phi = 0. \end{aligned} \quad (2)$$

Here, $\phi_0(\xi)$ plays the role of a potential that influences the motion of $\delta\phi$. By expanding $\delta\phi(\xi, \tau) = \sum_{k=1}^{\infty} b_k(\tau) \cos[k\xi + \alpha_k(\tau)]$, after $\{A_k, \theta_k\}$ are obtained from $\partial\phi_0/\partial\tau=0$, the modes $\{b_k(\tau), \alpha_k(\tau)\}$ can be solved from Eq. (2).

It is amazing that in this system with infinitely many dimensions the collision with the saddle point can be observed in a simple phase space. In Fig. 1 we plot the asymptotic attractor in the phase space ϕ vs $\partial\phi/\partial\xi$ at $\xi=0$ with (a) $\epsilon=0.18$, (b) $\epsilon=0.19$, (c) $\epsilon=0.20$, and (d) $\epsilon=0.2009 < \epsilon_c$; here $\phi(\xi, \tau) = \sum_{k=1}^N \{A_k \cos(k\xi + \theta_k) + b_k(\tau) \cos[k\xi + \alpha_k(\tau)]\}$ and $\partial\phi/\partial\xi = \sum_{k=1}^N k \{-A_k \sin(k\xi + \theta_k) - b_k(\tau) \sin[k\xi + \alpha_k(\tau)]\}$. In the plots (and in following Figs. 2, 4 as well) the saddle point $\phi_0^*(0)$ is denoted by a circle, respectively. Comparing Figs. 1(a)–1(d) one can see that when ϵ approaches a critical ϵ_c the attractor and the saddle point gradually approach each other. At $\epsilon=0.2009$, which is very close to ϵ_c , the attractor orbit almost (but not yet) touches the saddle point. In all these cases no crisis occurs and the attractors are in the SR state.

By increasing ϵ a little bit further, e.g., $\epsilon=0.2010$ in Fig. 2, the asymptotic attractor shows a completely different picture. Due to a collision with the saddle point, a crisis occurs, after which the basin of attraction of the attractor is greatly enlarged; in the new attractor the variation of $\partial\phi/\partial\xi$ with ϕ

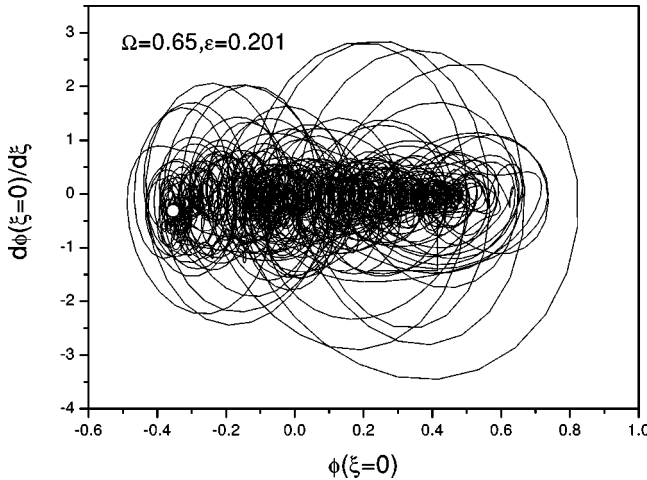


FIG. 2. Asymptotic STC attractor in the phase space $\phi(\xi=0)$ vs $\partial\phi(\xi=0)/\partial\xi$ for $\epsilon=0.2010$, with $\Omega=0.65$. “O” shows the saddle point (SSW).

is much more irregular. Our previous investigation shows that the new attractor is in the STC state for its spatial coherence has been destroyed. In this case, the saddle point gets embedded in the attractor.

Figure 3 shows examples of asymptotic contour plots of (a) a SR state for $\epsilon=0.18 < \epsilon_c$ and (b) a STC state for $\epsilon=0.21 > \epsilon_c$, in (a) the space-time behavior is slightly chaotic but a travelling wave can still be seen, in (b) the wave is broken and the spatial coherence no longer holds. The spatial spectrum in (a) shows an exponential law and in (b) a power law [9].

III. COLLISION WITH THE SADDLE POINT AT THE ONSET OF STC

For a given $\epsilon > \epsilon_c$, it is important to find out what happens at the onset of transition to the STC. From the results of Sec. II it is reasonable to anticipate an occurrence of collision with the saddle point in the temporal evolution. Figure 4 shows the transient attractor of $\partial\phi(0,\tau)/\partial\xi$ vs $\phi(0,\tau)$ for $\epsilon=0.22 > \epsilon_c$; (a)–(d) are for the same parameters but with different initial distributions $\phi(\xi,\tau=0)$, which are taken to be adjacent to the saddle point $\phi_0^*(\xi)$. To avoid confusion the orbit in the first few steps is not shown in the plot. In fact, in all these four examples (and in all the test runs) one can see the transient attractor colliding with the saddle point before the orbit transits to a much larger STC attractor.

The collision with the saddle point in Fig. 4 is consistent with the phenomenon of “pattern resonance” reported in Ref. [10]. We have pointed out that a pattern resonance is responsible for triggering a crisis of transition to the STC. As demonstrated by snapshots in Ref. [10], at the pattern resonance ($t=t_c$) the realized waveform $\phi(x,t_c)$ almost coincides with the virtual waveform of the SSW $\phi_0^*(x-\Omega t_c)$. Obviously in the phase space $\phi(\xi)$ vs $\partial\phi/\partial\xi$ a pattern resonance should be manifested as a collision with the saddle point. This is exactly what we observe in Fig. 4 at $\xi=0$.

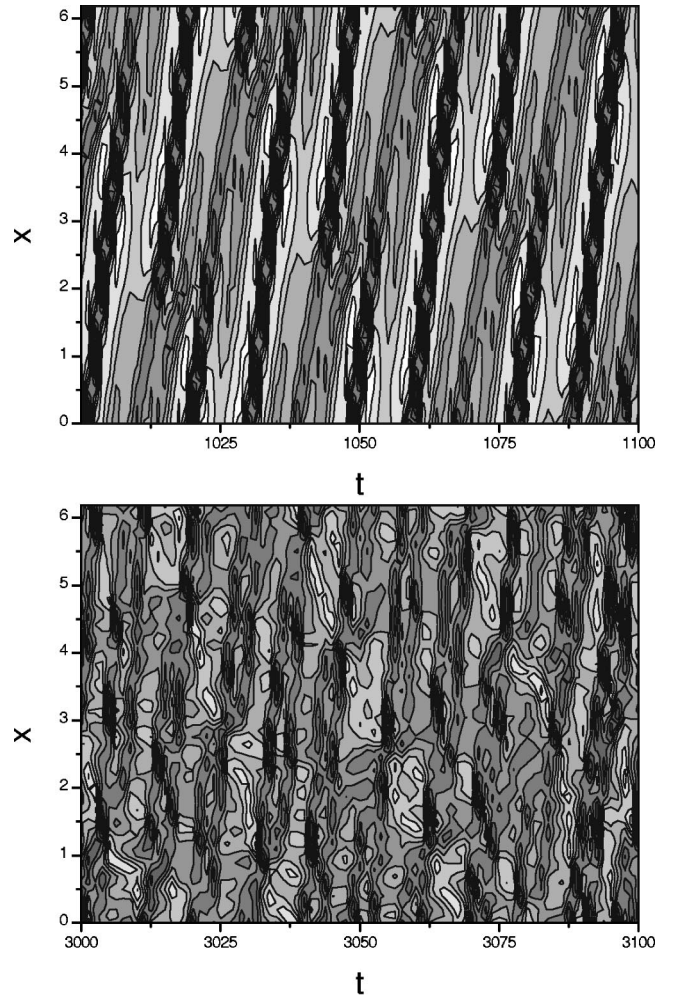


FIG. 3. Contour plot of $\phi(x,t)$ for (a) the SR attractor for $\epsilon=0.18$, and (b) the STC attractor for $\epsilon=0.21$.

Figure 5(a) is a contour plot of $d^2(x,t)$; here $d(x,t) \equiv \phi(x,t) - \phi_0^*(x - \Omega t)$, the difference between the realized wave and the SSW. In the plot an onset of crisis leading to transition from SR to STC occurs at $t=t_1 \approx 31$ as indicated by the arrow t_1 . Just around this moment one can find a zone with almost white color, indicating the waveform $\phi(x)$ is nearly the same as the virtual waveform $\phi_0^*(x)$ at $t=t_1$, i.e., the pattern resonance takes place. With the same initial condition Fig. 5(b) shows the variation of $\Delta(t) \equiv |\phi(x,t) - \phi_0^*(x - \Omega t)|$. Before transiting to the STC with higher level fluctuations one can identify a sharp spiky valley with an extremely small value of Δ at $t=t_1$ marked by the arrow t_1 , indicating the occurrence of the pattern resonance.

In the above plots, Figs. 3 and 5 are obtained by solving Eq. (1) with the pseudospectral method, and Figs. 1, 2, and 4 by solving $\phi_0(\xi)$ and $\delta\phi(\xi,\tau)$. It is of no surprise that they agree with each other qualitatively, since in deriving Eq. (2) no approximation has been made. All these results support that the pattern resonance or collision with the saddle point is the origin for the onset of the STC. In Ref. [10] we also pointed out that the pattern resonance is essentially due to a

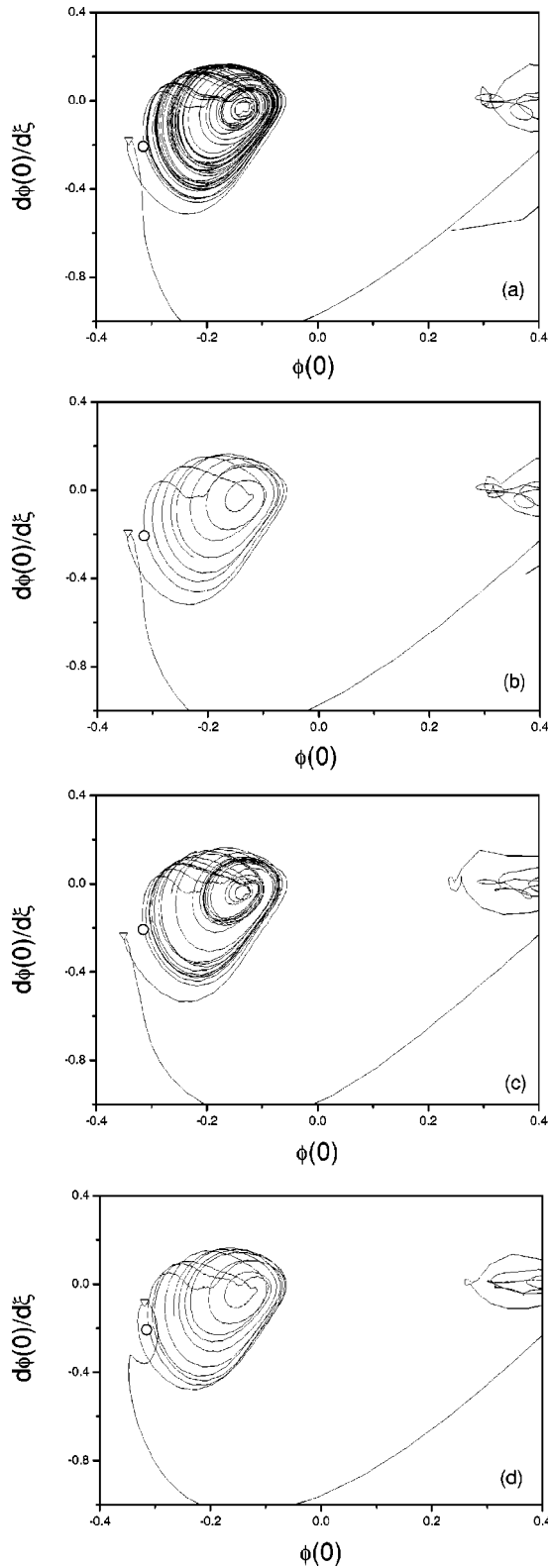


FIG. 4. Transient SR attractor and its collision with the saddle point in the phase space $\phi(\xi=0)$ vs $\partial\phi(\xi=0)/\partial\xi$ which leads to the crisis of transition to the STC: $\Omega=0.65$ and $\epsilon=0.22$. (a)–(d) are for different initial conditions. “○” and “▽” denote the saddle point (SSW) and the “ejecting point,” respectively.

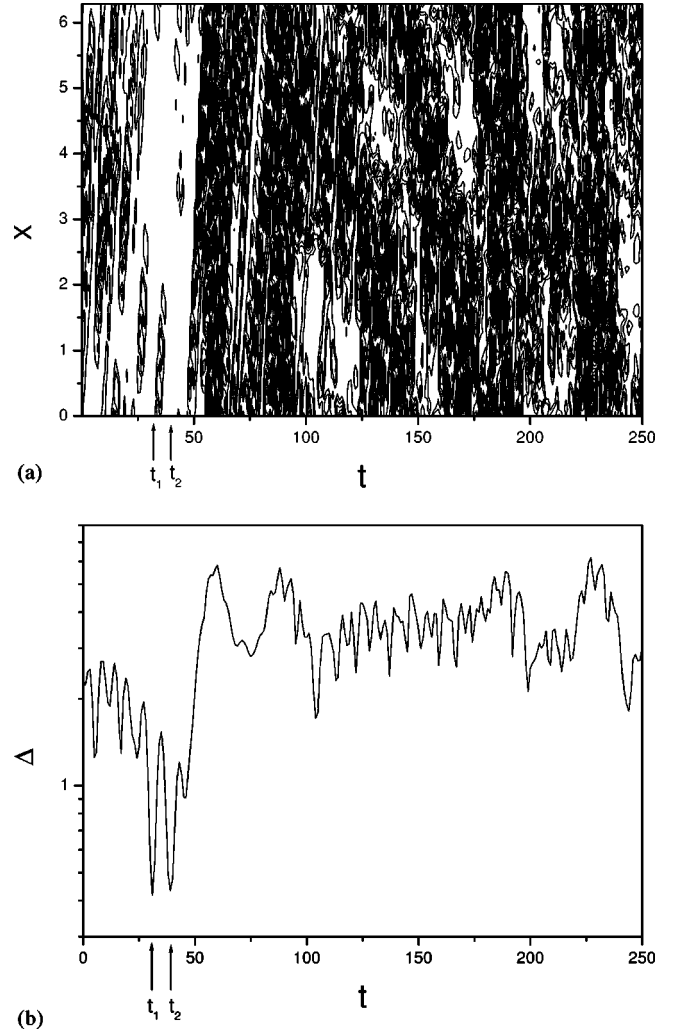


FIG. 5. (a) Contour plot of the squared distance $d^2(x,t)$ and (b) distance $\Delta(t)$ between the realized solution $\phi(x,t)$ and the corresponding SSW, $\phi_0^*(x-\Omega t)$: $\Omega=0.65$ and $\epsilon=0.22$. The initial conditions in (a) and (b) are the same. The arrows t_1 and t_2 mark the critical times corresponding to the sharp points of the two biggest spikes in (b), respectively.

nonlinear frequency resonance, which also applies to the collision with the saddle point discussed here.

IV. A SUBSEQUENT CRITICAL DYNAMIC EVENT AFTER THE COLLISION

There is evidence indicating that the collision with a saddle point is not the only critical event at the onset to the STC. As we noted in Ref. [10] [also see Fig. 5(b)], the biggest spike is always followed by a slightly smaller spike before transiting to the STC. On the other hand, in Fig. 4 after the collision the orbit is not immediately ejected to the new attractor; instead, it seems to continue moving smoothly for one more circle surrounding the old attractor; only when it once again approaches closer to the saddle point “(○),” does the orbit suddenly turn its direction and is ejected to the new attractor. In all tested runs we observed such an “ejecting point,” which is denoted by “▽” in Fig. 4. By compar-

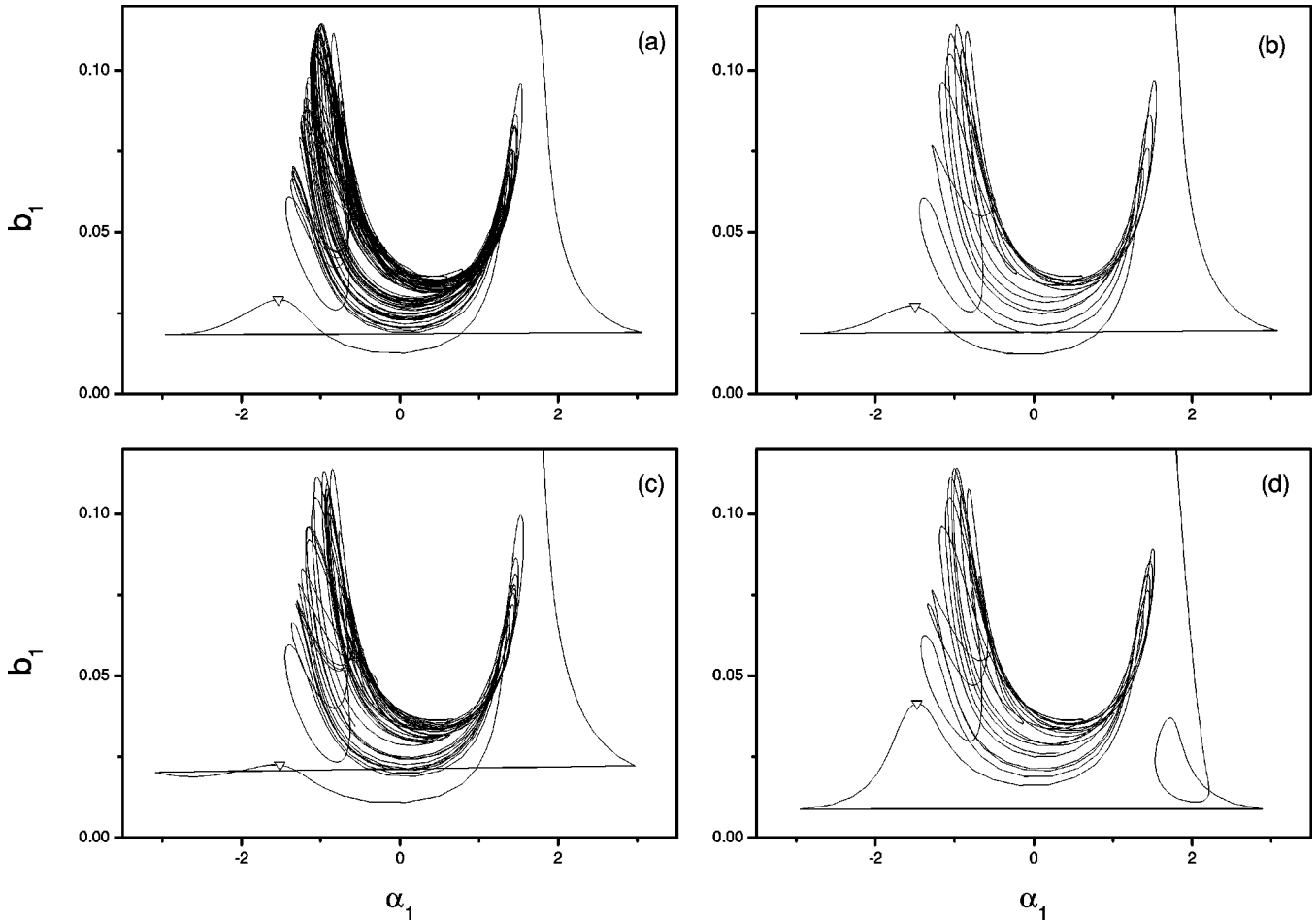


FIG. 6. $b_{k=1}(\tau)$ vs $\alpha_{k=1}(\tau)$ in the transient state. (a)–(d) correspond to the same cases as in Fig. 4; “ ∇ ” indicates the position of the “ejecting point.”

ing the behaviors in Figs. 5 and 4 qualitatively, it is evident that at the first big spike (marked by the arrow t_1) in Fig. 5 a collision to the saddle point takes places (see the colliding point \circ in Fig. 4), and near to the second big spike (marked by the arrow t_2) in Fig. 5 the orbit is ejected to the STC attractor (see the ejecting point ∇ in Fig. 4).

Investigating the behavior of the $k=1$ master mode may help us to understand what happens at the ejecting point. Figure 6 shows the transient orbit of $b_{k=1}$ vs $\alpha_{k=1}[\text{mod } (-\pi, \pi)]$ has been taken]; here (a)–(d) are for the same cases as in Figs. 4(a)–4(d), respectively. The corresponding position to the ejecting point is also denoted by ∇ , respectively, in Fig. 6. One can notice that they are all located at the top of a hump respectively. Most importantly, right after crossing over the hump, the variation of the mode phase $\alpha_{k=1}$ surpasses 2π , and the mode amplitude b_1 increases greatly, leading to the STC attractor. Conversely, before the ejecting point the phase $\alpha_{k=1}$ is confined within an angle less than 2π .

A significant phenomenon is that the $\alpha_{k=1}$ values at the ejecting points, α_1^* , in Figs. 6(a)–6(d) are nearly the same. For 20 test runs with different initial conditions we get the averaged value of $\alpha_1^* \approx -1.539$ with the averaged relative deviation $\langle |\alpha_1^* - \alpha_1^*| / \alpha_1^* \rangle \approx 0.039$. Therefore it is reasonable

to believe that near the ejecting point there exists a critical phase α_1^c , across which $\alpha_{k=1}$ experiences a state transition; i.e., before the point it is in vibration, beyond it $\alpha_{k=1}$ can be whirling as well as vibrating. This point is presumably a saddle, just as occurs when crossing the separatrix a vibrating pendulum becomes rotating. Figure 7 displays the time series of b_1 and α_1 for the same case as in Figs. 4(c) and 6(c). In the plot one can see that after the crisis (a) the fluctuation level of $b_{k=1}$ suddenly becomes very large and (b) $\alpha_{k=1}$ transits to a vibrating-whirling state. The arrow in Fig. 7(a) denotes the first critical dynamic event, i.e., the collision to the SSW, whereas the arrow in Fig. 7(b) denotes the second critical dynamic event, i.e., the state transition of $\alpha_{k=1}$. After the transition to the STC the master mode $\delta\phi_{k=1}(\xi, \tau) = b_1(\tau)\cos[k\xi + \alpha_1(\tau)]$ is no longer confined by the “potential” $\phi_0^*(\xi)$; its peak is allowed to move freely relative to it. This results in a wavebreaking of $\phi(x, t)$ and the destruction of spatial coherence.

V. CONCLUSION AND DISCUSSION

In the present work by using an appropriate representation in the driver frame it is demonstrated that a transition from weak chaos to the STC is induced by a collision with a

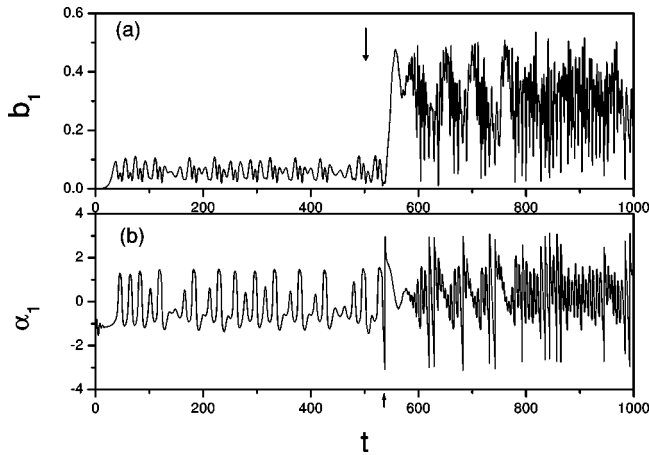


FIG. 7. Temporal evolutions of (a) $b_{k=1}$ and (b) $\alpha_{k=1}$. One can see a state transition taking place in (b). The same case as in Figs. 4(c) and 6(c).

saddle point (SSW). The result on the collision in the temporal evolution supports our previous observation that “pattern resonance” is responsible for driving the onset of a crisis to the STC. We also identify a dynamic event subsequent to the collision, at which the $k=1$ mode phase of the perturbation wave experiences a state transition. That is, we discover

that there are two critical dynamic events instead of one that are involved in the crisis of transition to the STC. The first event provides the possibility for the occurrence of the second one, but only after the second event is the wave broken.

In our case the collision with the saddle point (\odot) in Fig. 4 does not occur in a strict sense. At the moment of collision not all b_k are very close to zero, in particular for high- k modes the deviations from the saddle point can be apparent. This fact can also be seen in Figs. 5(a) and 5(b); at the pattern resonance marked by the arrow t_1 the realized $\phi(x, t_1)$ does not exactly coincide with the virtual $\phi_0^*(x, t_1)$. A small but finite difference between them can be seen. This is in contrast to the high-dimensional interior crisis observed in Ref. [8], where an exact collision with the saddle point is seen in every dimension. Moreover, in Ref. [8] the mode phase does not appear in the model, and hence no subsequent event involving state transition of the mode phase occurs following the collision. Presumably this is why its spatial coherence is still retained after the crisis.

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- [1] H.L. Swinney, *Physica D* **7**, 3 (1983).
 - [2] D. Biskamp and Kaifen He, *Phys. Fluids* **28**, 2172 (1985).
 - [3] T. Klinger, A. Latten, A. Piel, G. Bonhomme, T. Pierre, and T. Dudok de Wit, *Phys. Rev. Lett.* **79**, 3913 (1997).
 - [4] O. Grulke and T. Klinger, *New J. Phys.* **4**, 67.1 (2002).
 - [5] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, New York, 1993), p. 277.
 - [6] C. Grebogi, E. Ott, S. Pelikan, and J.A. Yorke, *Phys. Rev. Lett.* **48**, 1507 (1982).
 - [7] C. Grebogi, E. Ott, and J.A. Yorke, *Physica D* **7**, 181 (1983).
 - [8] A.C.-L. Chian, E.L. Rempel, E.E. Macau, R.R. Rosa, and F. Christiansen, *Phys. Rev. E* **65**, 035203 (2002).
 - [9] Kaifen He, *Phys. Rev. Lett.* **80**, 696 (1998).
 - [10] Kaifen He, *Phys. Rev. Lett.* **84**, 3290 (2000).